Yang–Mills vacuum structure and quantum gravity

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Abstract

Using renormalization group methods, we calculate the derivative expansion of the effective Lagrangian for a covariantly constant gauge field in curved spacetime. Curvature affects the vacuum; in particular it could induce phase transitions between different vacua. We also consider the effect of quantum fluctuations of the metric, in the context of a renormalizable R^2 theory. In this case the critical curvature depends on the gravitational coupling constants.

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1. Introduction

The one-loop effective Lagrangian for a gauge theory contains a term, called the chromomagnetic potential, describing the response of the system to a covariantly constant chromomagnetic field (by this we mean the space component of the field strength; the gauge field need not necessarily be the gluon. In fact this kind of analysis has been applied to electroweak theory.) It has been shown that the chromomagnetic potential can have a non zero minimum; the corresponding state is then called a chromomagnetic vacuum. It was also realized that such a state could not be the true vacuum of the theory: the effective potential has an imaginary part, due to the presence of an unstable field mode. There have been various proposals on how this state may be stabilized.

One may expect the presence of large chromomagnetic fields in the early universe, when spacetime curvature was not zero. Hence it seems natural to investigate the influence of (background) curvature effects, or even quantum gravity effects, on the chromomagnetic vacuum. In this paper we address this problem by calculating the chromomagnetic potential for a covariantly constant gauge field in curved spacetime, using renormalization group methods and the derivative expansion. In the case of an SU(2) gauge field, we show that curvature influences the position of the minimum of the chromomagnetic potential. There is the possibility of curvature–induced phase transitions from the nontrivial phase (for negative or positive but small curvature) to a phase with zero chromomagnetic field (for curvature above a certain positive critical value). The v.e.v. of the chromomagnetic field grows as curvature becomes more negative.

We then consider the contribution of a renormalizable R^2–type quantum gravity to the chromomagnetic potential. Again we will see that there is the possibility of having phase transitions, depending now on the background curvature as well as new parameters appearing in the effective action.

2. Renormalization group improved effective Lagrangian

It is well-known (see for example [1] or, for a general review, [2]) that the renormalizable Lagrangian for a nonabelian gauge theory in curved spacetime is given by

$$L = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + a_1 R^2 + a_2 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + a_3 G \ , \tag{1}$$

where $G^a_{\mu\nu} = \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + gf^a{}_{bc}A^b_\mu A^c_\nu$, ∇ is the spacetime covariant derivative, g is the gauge coupling constant, a_i are gravitational coupling constants, C is the Weyl tensor and G is the Gauss–Bonnet term.

This theory is known to be asymptotically free in flat spacetime [3] (for a discussion using the background field method see [4]). In curved spacetime the renormalization of g is the same as in flat space [1] and hence the theory is asymptotically free in the gauge coupling also in curved spacetime [1,2]. The last three terms represent the Lagrangian of the external gravitational field which should be added in order to have multiplicative renormalizability in curved spacetime [2].

Due to this multiplicative renormalizability we can easily derive the following renormalization group (RG) equation [2]:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_i\frac{\partial}{\partial\lambda_i} - \gamma_A A^a_\mu\frac{\delta}{\delta A^a_\mu}\right) L_{\text{eff}}(\mu, \lambda_i, A^a_\mu, g_{\mu\nu}) = 0 \ , \tag{2}$$

where $\lambda_i = (g, a_1, a_2, a_3)$, γ_A is the standard RG gamma function for the (background) field A^a_μ.

We will discuss the RG improved effective Lagrangian obtained as the direct solution of the RG equation. This technique is quite well-known from applications to the Coleman–Weinberg effective potential in flat [6] and curved [7] space, as well as the effective Lagrangian for a covariantly constant gauge field [5,8,10]. Note however that the usual electroweak vacuum becomes unstable for large magnetic field [8] (for a discussion of how stability may be restored in the electroweak theory see [9]).

Working in the background field method we will use the standard relation

$$\beta_g = g\gamma \tag{3}$$

and the one-loop beta function $\beta_g = -\frac{bg^3}{(4\pi)^2}$, $b > 0$, where the value of the coefficient b depends on the gauge group under consideration.

In the solution of the RG equation (2) it is convenient to represent the effective Lagrangian as follows:

$$L_{\text{eff}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} f_1(\mu, \lambda_i, A^a_\mu, g_{\mu\nu}) + L_{\text{ext}}(\mu, \lambda_i, A^a_\mu, g_{\mu\nu}) , \tag{4}$$

where f_1 is a dimensionless function, L_{ext} is the vacuum polarization (or quantum effective Lagrangian of the external fields) and the background gauge field in f_1 and L_{eff} always appears in some dimensionless combination.

Applying the RG equation to the effective action in the form (4) one can solve it as follows:

$$f_1(\mu, \lambda_i, A^a_\mu, g_{\mu\nu}) = \exp\left(-2\int_0^t \bar{\gamma}(g(t'))dt'\right) f_1(\mu e^t, \bar{\lambda}_i(t), A^a_\mu, g_{\mu\nu}) , \tag{5}$$

where $\frac{d\bar{\lambda}_i(t)}{dt} = \frac{\bar{\beta}_i(\bar{\lambda}_i(t))}{1+\bar{\gamma}(\bar{g}(t))}$, $\bar{\gamma} = \frac{\gamma}{1+\gamma}$ and the condition (3) has been used for the effective coupling. Below we will consider only the leading order approximation, where

$$\bar{\beta}_i = \frac{\beta_i}{1+\gamma} \approx \beta_i \quad ; \qquad\qquad \bar{\gamma} = \frac{\gamma}{1+\gamma} \approx \gamma \tag{6}$$

and the one loop beta and gamma functions should be used in (6). For L_{ext} we get

$$L_{\text{ext}}(\mu, \lambda_i, A^a_\mu, g_{\mu\nu}) = L_{\text{ext}}(\mu e^t, \lambda_i(t), A^a_\mu, g_{\mu\nu}) . \tag{7}$$

where $\frac{d\lambda_i}{dt} = \beta_i(\lambda(t))$.

Collecting (5) and (7) in (4) we get the RG improved effective Lagrangian for YM theory in curved spacetime.

Working in leading-log approximation, one should use some boundary condition to define the effective action at $t = 0$. As a boundary condition we will use the tree-level Lagrangian (1). As a result the RG improved effective action has the following form:

$$L = -\frac{1}{4}\frac{g^2(0)}{g^2(t)}G^a_{\mu\nu}G^{a\mu\nu} + a_1(t)R^2 + a_2(t)C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + a_3(t)G , \tag{8}$$

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where eqs. (3) and (6) have been used to rewrite f_1. The effective gravitational coupling constants are known for any renormalizable matter theory [2].

The next issue is the meaning of the RG parameter t. Usually t is taken as the logarithm of the effective mass appearing in the one-loop effective action. Another way (giving an equivalent result) is to use the one loop effective action as a boundary condition after solving the RG equation. Then t is defined by the condition of vanishing of the logarithmic term in the effective action.

The one-loop effective action in Yang–Mills theory in curved spacetime may be easily found in the background field method (in the gauge $\alpha = 1$):

$$\Gamma^{(1)} = \frac{1}{2} \ln \det \mathcal{D}_{\mu\nu}^{-1} - \ln \det(-\hat{\triangle}) , \tag{9}$$

where $A_\mu = gA_\mu^a T_a$, $G_{\mu\nu} = G_{\mu\nu}^a T_a$, $\hat{\nabla}$ is the gauge and spacetime covariant derivative, $\hat{\triangle} = \hat{\nabla}_\mu \hat{\nabla}^\mu$ and

$$\mathcal{D}_{\mu\nu} = -g_{\mu\nu}\hat{\triangle} + R_{\mu\nu} - 2G_{\mu\nu} . \tag{10}$$

Let us now turn to the discussion of the specific situation where the general formalism will be used.

3. The vacuum of SU(2) gauge theory in curved spacetime

We will now consider an SU(2) YM theory in curved spacetime. The determination of the covariantly constant field will be a straightforward generalization of the flat space condition:

$$\hat{\nabla}^{\mu ab} G_{\mu\nu}^b = 0 . \tag{11}$$

The solution of this equation may be found in a normal coordinate expansion where the first term will give the flat space solution and the remaining ones give curvature corrections. For the group SU(2) there are two invariant combinations of the curvatures:

$$F = \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} \quad ; \qquad Y = \frac{1}{4} G_{\mu\nu}^a G^{a*\mu\nu} , \tag{12}$$

so for the covariantly constant magnetic gauge field we get

$$F = \frac{1}{2} H^2 + O(R) . \tag{13}$$

Let us choose a background satisfying the condition $R_{\mu\nu} = \frac{1}{4} R g_{\mu\nu}$. From the results of refs.[5,8] we know that in flat space the effective mass is given by gH, hence in flat space $t = \frac{1}{2} \ln \frac{gH}{\mu^2}$ [5,8] (this can be seen by diagonalizing the mass matrix or by direct proper-time calculation).

At the same time we know that in curved spacetime with vanishing gauge field the effective mass is given by the curvature. Hence a possible choice of the RG parameter t is

$$t = \frac{1}{2} \ln \frac{\frac{R}{4} + gH}{\mu^2} . \tag{14}$$

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The RG improved effective Lagrangian for a magnetic field in curved spacetime is then

$$L_{\text{eff}} = -\frac{1}{2}\frac{g^2(0)}{g^2(t)}H^2 + a_1(t)R^2 + a_2(t)C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + a_3(t)G \ , \tag{15}$$

where

$$g^2(t) = \frac{g^2(0)}{1 + \frac{11g^2(0)t}{12\pi^2}} \ ; \qquad a_1(t) = a_1(0) \ ; \qquad a_3(t) = a_3(0) - \frac{62t}{120(4\pi)^2} \tag{16}$$

and $G = R^2/6$, $C_{\mu\nu\rho\sigma} = 0$ (due to the choice of background). Then

$$L_{\text{eff}} = -\frac{1}{2}\frac{g^2(0)}{g^2(t)}H^2 + \left(a_1 + \frac{1}{6}a_3 - \frac{62t}{720(4\pi)^2}\right)R^2 \ . \tag{17}$$

For g small, if next-to-leading terms of order g can be neglected, the minimum of the effective potential (which is minus of effective lagrangian) occurs for

$$gH_{\min} = \mu^2 \exp\left(-\frac{24\pi^2}{11g^2}\right) - R/4. \tag{18}$$

This solution generalizes the (unstable) chromomagnetic vacuum of ref.[5] due to the influence of the gravitational background. Note that unlike to above case the symmetry breaking for scalar potential in curved spacetime may occur more frequently (actually already on tree level) [11].

Furthermore we see that there is a curvature-induced phase transition with critical curvature

$$R_c = 4\mu^2 \exp\left(-\frac{24\pi^2}{11g^2}\right) \ . \tag{19}$$

For positive curvature above R_c the ground state of the theory is $H_{\min} = 0$. For curvatures below R_c the ground state has a novanishing background magnetic field. Negative curvatures increases the value of this background.

This picture is confirmed by numerical calculations for g not very small. For example choosing $g = 0.5$ and using the dimensionless variables $\frac{gH}{\mu^2} = \exp\left(-\frac{24\pi^2}{11g^2}\right)x$ and $\frac{R}{\mu^2} = \exp\left(-\frac{24\pi^2}{11g^2}\right)y$, for y=2.2, 1, 0, -0.5, the minimum occurs for x=0.21, 0.46, 0.6, 0.66 respectively. The critical curvature occurs for $y \approx 2.3$.

Note that we cannot study the region of large negative curvature due to the appearance of an imaginary part in the potential. It is possible that for some value of the curvature this imaginary part is cancelled by curvature effects. However to check this conjecture one would have to treat the external gravitational field exactly.

4. Vacuum of SU(2) gauge theory coupled to higher derivative gravity

Finally, we discuss the modifications which will appear in the presence of quantized gravity. For this we need a workable model of quantum gravity. We will work within the context of

higher derivative quantum gravity (for an introduction see [2]); it can be considered here as an effective theory of quantum gravity.

We will start from the Lagrangian written in the following form:

$$L = -\frac{1}{4}\frac{g^2(0)}{g^2(t)}G^a_{\mu\nu}G^{a\mu\nu} + \frac{1}{2\lambda}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \frac{\omega}{3\lambda}R^2 \ , \tag{20}$$

where λ and ω are gravitational coupling constants. The Gauss–Bonnet term can be neglected in this context.

Proceeding as in the previous section and using the fact that the renormalization of the gauge coupling is not affected by the new terms, we obtain the RG improved effective Lagrangian as

$$L = -\frac{1}{2}\frac{g^2(0)}{g^2(t)}H^2 - \frac{\omega(t)}{3\lambda(t)}R^2 \ , \tag{21}$$

where $g^2(t)$ is given again by (16) and

$$\lambda(t) = \frac{\lambda(0)}{1 + \frac{139\lambda(0)t}{10(4\pi)^2}}, \tag{22}$$

$$(4\pi)^2\frac{d\omega(t)}{dt} = -\frac{139}{10}\lambda(t)\omega(t) - \lambda(t)\left[\frac{10}{3}\omega^2(t) + 5\omega(t) + \frac{5}{12}\right] \ .$$

The gravitational effective couplings may be found in [2]. As mentioned above, there is no choice of t which eliminates the logarithms to all orders in a theory with many mass scales. Moreover, it is not realistically feasible to diagonalize the mass matrix in the theory with Lagrangian (1). Assuming that the magnetic field is of the same order of (or larger than) the curvature, in leading log approximation a reasonable choice for t will be

$$t = \frac{1}{2}\ln\frac{gH + cR}{\mu^2} \tag{23}$$

with c a constant of order one. We set $c = 1/4$ as in the preceding section.

The approximate effective Lagrangian may now be written as follows:

$$L_{\text{eff}} = -\frac{1}{2}\left(1 + \frac{11g^2t}{12\pi^2}\right)H^2 + \frac{1}{3(4\pi)^2}\left[\frac{10}{3}\omega^2 + 5\omega + \frac{5}{12}\right]tR^2 \ . \tag{24}$$

Numerical analyses show that the results are qualitatively the same as in the previous section. However, the value of the critical curvature depends on the coupling constants. For example, using again $g = 0.5$ for the values $\omega = -0.02$ and $\omega = -6$ (which are the approximate critical points of RG equation for ω) one finds critical curvature at $y = 2.1$ and $y = 0.2$ respectively. Hence, the phase diagram of the magnetic vacuum is affected by the initial values of the gravitational couplings.

5. Conclusions

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In summary, we have investigated the effective Lagrangian for a covariantly constant non-abelian gauge field in the presence of background and quantized gravitational field, using the RG equation and the adiabatic expansion. We have found that there are curvature-induced phase transitions between different vacua.

There are some obviuos extensions of the present work. First, it would be interesting to discuss these phenomena treating the gravitational field exactly (for example choosing a de Sitter background). That would give possibility to study the gravitational stability of magnetic vacuum.

Second, there may be interesting cosmological applications of this theory. In particular one may think of inflationary models without scalar fields where inflaton is composite vector.

Third, in realistic electroweak theory there is a stable ground state of the type considered here. The effect of curvature on the vacuum of this model will be considered elsewhere.

Finally, one may consider different Yang–Mills backgrounds, (for example instantons) and discuss the gravitational effects there.

Acknowledgements

Appendix

It is interesting to compare the results of an SU(2) gauge theory to those of QED. For QED we find instead of eq.(17):

$$L_{\text{eff}} = -\frac{1}{2}\frac{e^2(0)}{e^2(t)}H^2 + \frac{1}{6}\left(a_3 - \frac{73t}{360(4\pi)^2}\right)R^2 \ ,$$

where

$$e^2(t) = \frac{e^2(0)}{1 + \frac{e^2(0)t}{6\pi^2}} \ .$$

For zero curvature the maximum of the effective potential $V_{\text{eff}} = -L_{\text{eff}}$ is unreliably large and the minimum is for $H = 0$. In general, however, curvature can influence the position of both stationary points.

References

1. I.L. Buchbinder and S.D. Odintsov, Izw. VUZov, Fiz. (Soviet Physics Journal) **n4** (1983) 46; *ibid.* **n12** (1983) 108; Yad. Fiz. (Sov. J. Nucl. Phys.) **40** (1984) 1338; Lett. Nuovo Cim. **42** (1985) 379.
2. I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, "Effective Action in Quantum Gravity", IOP Publishing, Bristol and Philadelphia (1992).
3. D. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343; D. Politzer, Phys. Rev. Lett. **30** (1973) 1346.
4. G. 't Hooft, Nucl. Phys. **B 62** (1973) 444.

5. G. Savvidy, Phys. Lett. **B 71** (1977) 133;
   S. Matinyan and G. Savvidy, Nucl. Phys. **B 134** (1978) 539.
6. S. Coleman and E. Weinberg, Phys. Rev. **D 7** (1973)1888;
   C. Ford, D.R.T. Jones, P.W. Stephenson and M. Einhorn, Nucl. Phys. **B 395** (1993) 17.
7. E. Elizalde and S.D. Odintsov, Phys. Lett. **B 303** (1993) 240; *ibid.* **B 321** (1994) 199;
   E. Elizalde, A. Romeo and S.D. Odintsov, Phys. Rev. **D 51** (1995) 1680.
8. N.K. Nielsen and P.Olesen, Nucl. Phys. **B 144** (1978) 376;
   J. Ambjørn, R.J. Hughes and N.K. Nielsen, Ann. Phys. (NY) **150** (1983) 92; H.B. Nielsen and P.Olesen, Nucl. Phys. **B 160** (1979) 380;
   J. Ambjørn and R.J. Hughes, Nucl. Phys. **B 197** (1982) 113.
9. J. Ambjørn and P. Olesen, Nucl. Phys. **B 330** (1990) 193.
10. E. Elizalde, Nucl. Phys. **B 243** (1984) 398.
11. G. Shore, Ann. Phys. **128** (1980) 376.